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Three-dimensional linear analysis for composite axially symmetrical circular plates

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Abstract

After constructing some elasticity models, a set of close three-dimensional linear analytical solutions, taking account of all of the normal stresses, shear stresses and satisfying all the equations of equilibrium, the mid-plane clamped boundary conditions and interfacial continuity conditions through-thickness, are presented for axially symmetrical homogeneous isotropic circular plates, laminates and sandwich plates under uniform transverse load by using the variable-separating method and formulating a set of displacement functions. Reasonability of the present solutions is demonstrated comparing with FEM analysis, "pb-2 Ritz" theory analysis and experimental results of sandwich plates.

Keywords: Isotropic; Sandwich plates; Laminates; Three-dimensional linear analysis; Clamped boundary

1. Introduction

Notableness for high ratio of performance/weight and designable performance, and increasing utilization of composite plates in various branches of engineering, has led to increasing research activities in mechanical characteristics, structural modeling. Two-dimensional theory literature related to composite plates is abundant. However, these theories based on Kirchhoff's assumptions have often neglected effects of shear stresses and normal stress σ_z, which is inadequate in accurately estimating for mid-plates and thick plates.

Timoshenko and Goodier (1970) presented a set of three-dimensional solutions for axially symmetrical homogeneous isotropic circular plates in "Theory of Elasticity":

$$\sigma_r = q_0\left[\frac{2+v}{8}\frac{z^3}{h^3} - \frac{3(3+v)}{32}\frac{r^2 z}{h^3} - \frac{3}{8}\frac{z}{h}\right], \quad \sigma_z = q_0\left(-\frac{z^3}{4h^3} + \frac{3}{4}\frac{z}{h} - \frac{1}{2}\right), \quad \tau_{rz} = -\frac{3q_0 r}{8h^3}(h^2 - z^2)$$

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The solutions cannot satisfy some displacement boundary conditions, and it is evident that the stress σ_r at the mid-surface vanishes.

A high-order theory of plate deformation, supposing the three-dimensional displacement field is a linear combination of thickness co-ordinate, was proposed by Lo et al. (1977) and shown to yield very accurate results, especially when the ratio of length-thickness is larger than 4, it is in aggreement with accurately elastic mechanical solutions. However, complex and expensive computation impedes wide application of LCW high-order theory. Noting this restriction of the traditional plate and shell theories, Reddy (1984) proposed a simplified higher-order theory, assuming the thickness to be incompressible, with very accurate for displacements and normal stresses but not for shear stresses.

Liew (1992) have introduced "pb-2" functions which is combination of a shape function and a polynomial function into the plate theory of Reddy (1984) for predicting the deflections, bending moments, buckling loads, and vibration frequencies of plates. The method has been shown to be computationally efficient and numerically accurate for the analysis of plates with complex geometry and arbitrary support conditions. However, it still belongs to two-dimensional theory and is difficult to satisfy stress boundary conditions. Assuming that the strains in plane are functions of the thickness co-ordinate, z and the strains out plane independent on z, Liew (1994, 1995), Liew and Yang (1999, 2000) have made free vibration analysis of clamped circular laminates, thick plates and annular plates using "pb-2 Ritz" method, and have demonstrated to be accurate comparing with theory solutions published and FEA analysis.

A sandwich plate generally comprises two faceplates and a flat plate made up of a relatively-thick core with low density material, and different from laminate. There exist a vast theoretical and numeric analysis literature, such as deflection and buckling model for sandwich plates presented by Wang (1995a,b), nonlinear vibration for circular sandwich plates studied by Du and Li (2000). Frostig et al. (1992) proposed a high-order buckling theory for sandwich beams, Barut et al. (2001) made analysis of thick sandwich construction by a $\{3,2\}$-order theory, but Roberts et al. (1998) demonstrated poor agreement between theoretical, FEA results and experimental results.

In this paper we formulate the elastic solutions for axially symmetrical composite circular plates including single layer, laminate and sandwich plates under transverse uniform static loading. The aim of the present work is to provide a benchmark for approximate solutions. The theoretical analysis is organized as follows: first, an exact analysis of axially symmetrical homogeneous isotropic circular plates is attempted and the close three-dimensional linear solutions are obtained by separating the variables; then by adopting the functions of homogeneous isotropic circular plates, analysis for axially symmetrical sandwich and laminated plates subjected to the boundary conditions and the interfacial continuity conditions is attempted.

The laminate considered herein is made of layers of homogeneous isotropic materials with similar properties, and clamped on the edge of mid-plane. The transverse stresses and the displacements are assumed continuous through the laminate thickness.

In the present paper, the sandwich plate, clamped on the edges of the mid-planes of the face plates, consists of two or three different homogeneous isotropic materials elastically bonded together, with large differences of density and modulus between the face plates and the core. The transverse stresses are assumed continuous through the thickness, but the displacements continuous through the thickness only at the center.

To verify the present solutions, "pb-2 Ritz" models for single layer and laminate under uniform static loading are set up in Section 3, and experiment of sandwich plates presented in Section 4. Subsequently, the present theoretical calculation results are compared with numerical, "pb-2 Ritz" theoretical and experimental results.

2. Theoretical analysis

2.1. Three-dimensional analytical formulation for axially symmetrical homogeneous isotropic circular plates

The axially symmetrical homogeneous isotropic circular plate under uniform transverse load as shown in Fig. 1 is made of an isotropic material with Young's modulus E and Poisson's ratio v. U, V and W stand for the displacements in the radial, circumferential, and axial directions, respectively.

The following non-dimensional parameters are introduced:

$$r = \frac{R}{a}, \quad z = \frac{Z}{a}, \quad u = \frac{U}{a}, \quad v = \frac{V}{a}, \quad w = \frac{W}{a}, \quad h = \frac{H}{a}$$

The deformations are symmetrical about z-axis, thus the stresses and the strains are independent of θ and

$$\tau_{r\theta} = \tau_{z\theta} = 0, \quad v = 0$$

While neglecting the gravity, we obtain the three-dimensional equations of equilibrium for the plate in the form

$$\begin{cases} \dfrac{e_{,r}}{1-2v} + \nabla^2 u - \dfrac{u}{r^2} = 0 \\ \dfrac{e_{,z}}{1-2v} + \nabla^2 w = 0 \end{cases} \tag{1}$$

where a comma denotes partial differentiation with respect to the suffix variable (the followings are the same), and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad e = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$$

The geometric equations are

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} u_{,r} \\ \dfrac{u}{r} \\ w_{,z} \\ u_{,z} + w_{,r} \end{Bmatrix}$$

The stresses–strains relations are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{Bmatrix} = \frac{Eve}{1-v-2v^2} + \frac{E}{1+v}\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{Bmatrix}, \quad \tau_{rz} = \frac{E\gamma_{rz}}{2(1+v)}$$

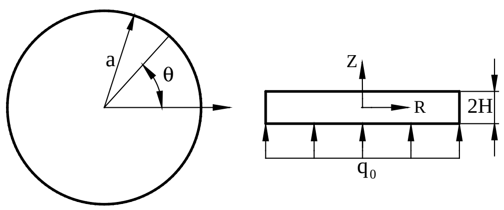

Fig. 1. Geometry of a circular plate under uniform transverse load.

The boundary conditions of uniform transverse pressure applied at the surface are

$$
\begin{aligned}
&z = h, \quad \tau_{rz} = 0, \quad \sigma_z = 0 \\
&z = -h, \quad \tau_{rz} = 0, \quad \sigma_z = -q_0 \\
&r = 0, \quad u = w_{,r} = 0 \\
&r = 1, \quad \int_{-h}^{h} \tau_{rz}\, dz = -\frac{q_0}{2} \\
&z = 0, \quad r = 1, \quad u = w = w_{,r} = 0
\end{aligned}
$$

Because of the symmetry of the construction, the boundary conditions with respect to the z-axis, the deformations are also symmetrical, i.e., $w_{(x,y,z)} = w_{(-x,y,z)}$, $u_{(x,y,z)} = -u_{(-x,y,z)}$ in an orthogonal coordinate system where the z axis coincides with the center. This means that u is an odd function and w an even function with respect to the coordinate x. While the coordinate r coincides with the coordinate x in the positive direction, $w_{(x,0,z)} = w_{(r,z)}$, $u_{(x,0,z)} = u_{(r,z)}$, so u is an odd function and w an even function with respect to the coordinate r. Let us now take

$$
\begin{cases} u = a_1 r + a_3 r^3 \\ w = b_0 + b_2 r^2 + b_4 r^4 \end{cases} \tag{2}
$$

Substituting Eq. (2) in Eq. (1), we have

$$
\left(16a_3 \frac{1-v}{1-2v} + 2b_{2,z}\frac{1}{1-2v} + a_{1,zz}\right) r + \left(4b_{4,z}\frac{1}{1-2v} + a_{3,zz}\right) r^3 = 0
$$

$$
2a_{1,z}\frac{1}{1-2v} + 4b_2 + 2b_{0,zz}\frac{1-v}{1-2v} + \left(4a_{3,z}\frac{1}{1-2v} + 16b_4 + 2b_{2,zz}\frac{1-v}{1-2v}\right) r^2 + 2b_{4,zz}\frac{1-v}{1-2v} r^4 = 0
$$

then

$$
\begin{aligned}
&b_{4,zz} = 0 \\
&4b_{4,z}\frac{1}{1-2v} + a_{3,zz} = 0 \\
&2a_{3,z}\frac{1}{1-2v} + 8b_4 + b_{2,zz}\frac{1-v}{1-2v} = 0 \\
&16a_3\frac{1-v}{1-2v} + 2b_{2,z}\frac{1}{1-2v} + a_{1,zz} = 0 \\
&a_{1,z}\frac{1}{1-2v} + 2b_2 + b_{0,zz}\frac{1-v}{1-2v} = 0
\end{aligned}
$$

taking account of the boundary conditions, we can get

$$
\begin{aligned}
u &= c_0 r z^3 \frac{16(2-v)}{3(1-v)} + c_{03} z r + c_{04} r - 4c_0 z r^3 \\
w &= -c_0 z^4 \frac{8(1+v)}{3(1-v)} - c_{03} z^2 \frac{1}{2(1-v)} - c_{02} z^2 \frac{1-2v}{1-v} + c_{05} z + \frac{8c_0 v z^2 r^2}{1-v} + c_2 r^2 + c_0 r^4 + c_{06}
\end{aligned} \tag{3}
$$

where

$$
c_0 = 3q_0\frac{(1-v^2)}{128Eh^3}, \quad c_{02} = -2c_0, \quad c_{03} = -32c_0 h^2 \frac{1}{1-v} + 4c_0, \quad c_{04} = 0,
$$

$$
c_{05} = -c_0 \frac{64(1-2v)h^3}{3(1-v)^2}, \quad c_{06} = c_0
$$

Substituting Eq. (3) in the geometric equations and the equations of the stresses-strains relations, we will obtain solutions of the stresses

$$\begin{aligned}
\sigma_r &= q_0\left[\frac{2+v}{8}\frac{z^3}{h^3} - \frac{3(3+v)}{32}\frac{r^2 z}{h^3} + \frac{3}{32}\frac{z}{h^3}\left(1+v-\frac{8h^2}{1-nu}\right) - \frac{v}{2(1-v)}\right] \\
\sigma_\theta &= q_0\left[\frac{2+v}{8}\frac{z^3}{h^3} - \frac{3(1+3v)}{32}\frac{r^2 z}{h^3} + \frac{3}{32}\frac{z}{h^3}\left(1+v-\frac{8h^2}{1-v}\right) - \frac{v}{2(1-v)}\right] \\
\sigma_z &= q_0\left(-\frac{z^3}{4h^3} + \frac{3}{4}\frac{z}{h} - \frac{1}{2}\right) \\
\tau_{rz} &= -\frac{3q_0 r}{8h^3}(h^2 - z^2)
\end{aligned} \tag{4}$$

The last two of Eq. (4) coincide with the ones presented by Timoshenko and Goodier, however, the third and fourth terms in the expression of σ_r are different. From Eq. (4) we can find:

$$\sigma_r = \sigma_\theta = -\frac{q_0 v}{2(1-v)} \quad \text{for} \quad z = 0$$

It should be noted that the value of $\frac{q_0 v}{2(1-v)}$ is a constant throughout the body and not a small quantity comparing with the other terms, provided a thick plate is loaded.

While the plate is thin enough, i.e., $z \to 0$, it is easy to see from Eq. (3) that w is of identical formula with the two-dimensional solution of Timoshenko and Woinowsky (1959).

$$w = q_0(1-r^2)^2\frac{3(1-v^2)}{128Eh^3}$$

2.2. Three-dimensional analytical formulation for sandwich axially symmetrical isotropic circular plates

For convenience, we use suffix ‘t’, ‘c’, ‘b’ to denote the top, the core and the bottom plates.

The sandwich axially symmetrical circular plate under uniform transverse loading is as shown in Fig. 2. Each phase is made of isotropic material with Young’s modulus E_i and Poisson’s ratio v_i. U_i, V_i and W_i respectively stand for the displacements in the radial, circumferential, and axial directions. Because the deformations are symmetrical about z-axis, the stresses and the strains are independent of θ, we have

$$\tau_{r\theta i} = \tau_{z\theta i} = 0, \quad v_i = 0 \ \ (i = \mathrm{t, c, b})$$

The following non-dimensional parameters are introduced

$$r = \frac{R}{a}, \quad z = \frac{Z}{a}, \quad u_i = \frac{U_i}{a}, \quad w_i = \frac{W_i}{a}, \quad h = \frac{H}{a}, \quad h_0 = \frac{H_0}{a}$$

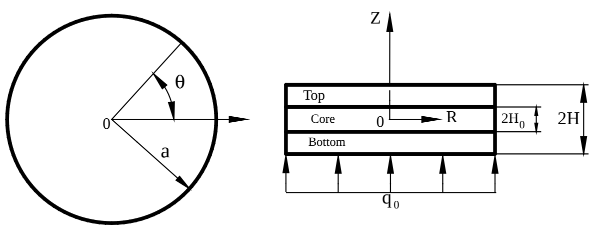

Fig. 2. Geometry of a sandwich plate under uniform transverse load.

When neglecting the gravity, the three-dimensional equations of equilibrium, geometric equations and stresses–strains relations are similar to the ones for the axially symmetrical homogeneous isotropic circular plate, then, for each layer, the displacement solutions are

$$
\begin{aligned}
u_i &= c_{i0} r \bar{z}^3 \frac{16(2-v_i)}{3(1-v_i)} + c_{i03}\bar{z}r + c_{i04}r - 4c_{i0}\bar{z}r^3 \\
w_i &= -c_{i0}\bar{z}^4 \frac{8(1+v_i)}{3(1-v_i)} - c_{i03}\bar{z}^2 \frac{1}{2(1-v_i)} - c_{i02}\bar{z}^2 \frac{1-2v_i}{1-v_i} + c_{i05}\bar{z} + \frac{8c_{i0}v_i\bar{z}^2r^2}{1-v_i} + c_{i2}r^2 + c_{i0}r^4 + c_{i06}
\end{aligned} \tag{5}
$$

where

$$
\bar{z} = \begin{cases} z - \dfrac{h+h_0}{2} & \text{if } z \geqslant h_0 \\ z & \text{if } |z| \leqslant h_0 \\ z + \dfrac{h+h_0}{2} & \text{if } z \leqslant -h_0 \end{cases}
$$

The boundary conditions are

$$
\begin{aligned}
& z = h, \quad \tau_{rz(\mathrm{t})} = 0, \quad \sigma_{z(\mathrm{t})} = 0 \\
& z = -h, \quad \tau_{rz(\mathrm{b})} = 0, \quad \sigma_{z(\mathrm{b})} = -q_0 \\
& r = 0, \quad u_i = w_{i,r} = 0 \\
& z = \frac{h+h_0}{2}, \; r = 1, \quad u_\mathrm{t} = w_\mathrm{t} = w_{\mathrm{t},r} = 0 \\
& z = 0, \quad u_\mathrm{c} = w_\mathrm{c} = w_{\mathrm{c},r} = 0 \\
& z = -\frac{h+h_0}{2}, \; r = 1, \quad u_\mathrm{b} = w_\mathrm{b} = w_{\mathrm{b},r} = 0 \\
& r = 1, \quad \int_{-h}^{h} \tau_{rz}\,\mathrm{d}z = -\frac{q_0}{2}
\end{aligned}
$$

and interfacial continuity conditions are

$$
\begin{aligned}
& z = h_0, \quad \sigma_{z(\mathrm{t})} = \sigma_{z(\mathrm{c})}, \tau_{rz(\mathrm{t})} = \tau_{rz(\mathrm{c})} \\
& z = h_0, \quad r = 0, \quad w_\mathrm{t} = w_\mathrm{c}, u_\mathrm{t} = u_\mathrm{c} \\
& z = -h_0, \quad \sigma_{z(\mathrm{b})} = \sigma_{z(\mathrm{c})}, \quad \tau_{rz(\mathrm{b})} = \tau_{rz(\mathrm{c})} \\
& z = -h_0, \quad r = 0, \quad w_\mathrm{b} = w_\mathrm{c}, \quad u_\mathrm{b} = u_\mathrm{c}
\end{aligned}
$$

For the sake of simplicity, assuming

$$
\begin{aligned}
& z = h_0, \quad \sigma_z = -q_2 \\
& z = -h_0, \quad \sigma_z = -q_1
\end{aligned}
$$

Substituting the displacement functions in the boundary conditions we obtain

For the top face plate

$$
\begin{aligned}
& c_{0(\mathrm{t})} = 3q_2 \frac{1-v_\mathrm{t}^2}{128E_\mathrm{t}\bar{h}^3}, \quad c_{2(\mathrm{t})} = -2c_{0(\mathrm{t})}, \quad c_{3(\mathrm{t})} = -32c_{0(\mathrm{t})}\bar{h}^2 \frac{1}{1-v_\mathrm{t}} + 4c_{0(\mathrm{t})} \quad c_{4(\mathrm{t})} = 0, \\
& c_{5(\mathrm{t})} = -c_{0(\mathrm{t})} \frac{64(1-2v_\mathrm{t})\bar{h}^3}{3(1-v_\mathrm{t})^2}, \quad c_{6(\mathrm{t})} = c_{0(\mathrm{t})}
\end{aligned}
$$

For the bottom face plate

$$c_{0(\mathrm{b})} = 3(q_0 - q_1)\frac{1 - v_\mathrm{b}^2}{128E_\mathrm{b}\bar{h}^3}, \quad c_{2(\mathrm{b})} = -2c_{0(\mathrm{b})},$$

$$c_{3(\mathrm{b})} = -32c_{0(\mathrm{b})}\bar{h}^2\frac{1}{1 - v_\mathrm{b}} + 4c_{0(\mathrm{b})} \quad c_{4(\mathrm{b})} = 0, \quad c_{5(\mathrm{b})} = -c_{0(\mathrm{b})}\frac{64(1 - 2v_\mathrm{b})\bar{h}^3}{3(1 - v_\mathrm{b})^2}, \quad c_{6(\mathrm{b})} = c_{0(\mathrm{b})}$$

where

$$\bar{h} = \frac{h - h_0}{2}$$

Employment in the constitutive equations of the face plates results in

$$z = h_0, \quad \tau_{rz(\mathrm{t})} = 0$$
$$z = -h_0, \quad \tau_{rz(\mathrm{b})} = 0$$

$$r = 1,$$
$$\int_{h_0}^{h} \tau_{rz}\,\mathrm{d}z + \int_{-h}^{-h_0} \tau_{rz}\,\mathrm{d}z = \frac{8}{3}(h - h_0)^3\left(\frac{c_{0(\mathrm{t})}E_\mathrm{t}}{1 - v_\mathrm{t}} + \frac{c_{0(\mathrm{b})}E_\mathrm{b}}{1 - v_\mathrm{b}}\right) + \left(\left(2c_{2(\mathrm{t})} + c_{3(\mathrm{t})}\right)E_\mathrm{t} + \left(2c_{2(\mathrm{b})} + c_{3(\mathrm{b})}\right)E_\mathrm{b}\right)(h - h_0)$$

Since the constants of the top and the bottom face plates are linear functions of q_2, $q_0 - q_1$, respectively, the displacement w_i at the center can be expressed in a simple form as

$$w_\mathrm{t} = Q_2 \ * q_2, \quad w_\mathrm{b} = Q_1 \ * \ (q_0 - q_1)$$

Fulfillment the core plate of the boundary and interfacial continuity conditions, the constants of the core can be obtained

$$c_{0(\mathrm{c})} = \frac{3(1 - v_\mathrm{c})(q_1 - q_2)}{128E_\mathrm{c}h_0^3},$$

$$c_{2(\mathrm{c})} = \frac{(5 + v_\mathrm{c}^2)h_0(q_2 - q_1) + 8E_\mathrm{c}(1 - v_\mathrm{c})(Q_2q_2 + Q_1q_0 - Q_1q_1)}{32h_0^2v_\mathrm{c}E_\mathrm{c}}$$

$$c_{3(\mathrm{c})} = \frac{(5 - 12v_\mathrm{c} + v_\mathrm{c}^2)h_0(q_1 - q_2) - 8E_\mathrm{c}(1 - v_\mathrm{c})(Q_2q_2 + Q_1q_0 - Q_1q_1)}{16h_0^2v_\mathrm{c}E_\mathrm{c}}$$

$$c_{\mathrm{c}04} = -\frac{(Q_2q_2 - Q_1q_0 + Q_1q_1)E_\mathrm{c}(1 - v_\mathrm{c}) + h_0(q_1 + q_2)(1 - 2v_\mathrm{c})}{4h_0E_\mathrm{c}v_\mathrm{c}}$$

$$c_{5(\mathrm{c})} = \frac{Q_2q_2 - Q_1q_0 + Q_1q_1}{2h_0}$$

$$c_{6(\mathrm{c})} = -c_{0(\mathrm{c})} - c_{2(\mathrm{c})}$$

Further

$$c_{2(\mathrm{c})} = -2c_{0(\mathrm{c})} \tag{6}$$

From the constitutive relations, we get

$$r = 1, \quad \int_{-h_0}^{h_0} \tau_{rz}\,\mathrm{d}z = \frac{8}{3}h_0^3\frac{c_{0(\mathrm{c})}E_\mathrm{c}}{1 - v_\mathrm{c}} + (2c_{2(\mathrm{c})} + c_{3(\mathrm{c})})E_\mathrm{c}h_0$$

It is easy to see that $c_{i(\mathrm{c})}$ and as $r = 1$, $\int_{h_0}^{h} \tau_{rz}\,\mathrm{d}z$, $\int_{-h}^{-h_0} \tau_{rz}\,\mathrm{d}z$, $\int_{-h_0}^{h_0} \tau_{rz}\,\mathrm{d}z$ are linear functions of q_0, q_1, q_2, so simple forms can be got

$$r = 1, \quad \int_{-h}^{h} \tau_{rz}\,dz = A_1 q_0 + A_2 q_1 + A_3 q_2, \tag{7}$$

$$c_{2(c)} + 2c_{0(c)} = B_1 q_0 + B_2 q_1 + B_3 q_2 \tag{8}$$

Substituting Eqs. (7) and (8) into the equations of boundary conditions and Eq. (6), we have

$$q_1 = \frac{2B_1A_3 - (2A_1 + 1)B_3}{2(B_3A_2 - A_3B_2)} q_0$$
$$q_2 = \frac{2B_1A_2 - (2A_1 + 1)B_2}{2(B_2A_3 - A_2B_3)} q_0$$

When the sandwich plate is symmetrical construction with respect to the mid-plane, $q_0 \approx q_1 + q_2$.

2.3. Three-dimensional analytical formulation for multilayered axially symmetrical isotropic circular plates

As shown in Fig. 3, the laminate is axially symmetrical plate under uniform transverse load with $(2N + 1)$ layers, total thickness $2H$. The ith layer $(i = 0, 1, 2, \ldots, n)$ is made of material with Young's moduli E_i, Poison's ratio v_i and thickness $2H_i$. The mid-plane of the 0th layer coincides with the global mid-surface. U_i, V_i and W_i respectively stand for the displacements in the radial, circumferential, and axial directions. The same non-dimensional parameters as above are introduced.

While neglecting the gravity, the three-dimension equations of equilibrium, geometric equations and stresses-strains relations are similar to the ones for the axially symmetrical homogeneous isotropic circular plate, then, for each layer, the displacement solutions are the similar to Eq. (5), just replacing the $\bar{z}$ with z. Then the boundary conditions are

$$z = h, \quad \tau_{rz(N)} = 0, \quad \sigma_{z(N)} = 0$$
$$z = -h, \quad \tau_{rz(-N)} = 0, \quad \sigma_{z(-N)} = -q_0$$
$$r = 0, \quad u_i = w_{i,r} = 0$$
$$r = 1, \quad \int_{-h}^{h} \tau_{rz(i)}\,dz = -\frac{q_0}{2}$$
$$z = 0, \ r = 1, \quad u_i = w_i = w_{i,r} = 0$$

and the interfacial continuity conditions

$$z = -h_0 + \sum_{m=0}^{i} 2h_m, \begin{cases} \sigma_{z(i)} = \sigma_{z(i+1)}, \tau_{rz(i)} = \tau_{rz(i+1)} \\ u_i = u_{(i+1)}, w_i = w_{(i+1)} \end{cases}$$

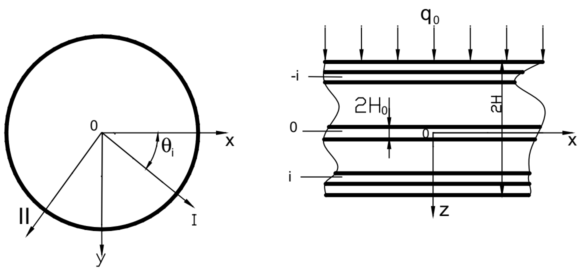

Fig. 3. Geometry of a laminate under uniform transverse load.

From the continuity conditions of the displacements, it is easy to see that all of the constants $c_{0(i)}$ through the thickness are the same one. Now take

$$c_{0(i)} = c_0$$

From the boundary conditions we have

$$c_{4(0)} = 0, \quad c_{2(0)} = -2c_0, \quad c_{6(0)} = c_0$$

To satisfy the continuity of the displacement w at layer interfaces, we must have

$$c_{2(i)} = c_{2(-i)} = -2c_0 - 8c_0 \sum_{k=1}^{i} \left(\frac{v_k}{1 - v_k} - \frac{v_{k-1}}{1 - v_{k-1}} \right) \left(-h_0 + \sum_{m=0}^{k-1} 2h_m \right)^2$$

Vanishing of the transverse shear stresses on the top and bottom and fulfilling of the continuity conditions result in

$$c_{3(N)} = -\frac{32c_0}{1 - v_N} h^2 - 2c_{2(N)}$$

$$\begin{aligned} c_{3(i)} &= c_{3(-i)} \\ &= \frac{32c_0E_{i+1}}{E_i} \left(\frac{1}{1 - v_{i+1}} \left(h - \sum_{m=N}^{i+1} 2h_m \right)^2 + 2c_{2(i+1)} + c_{3(i+1)} \right) - \frac{32c_0}{1 - v_i} \left(h - \sum_{m=N}^{i+1} 2h_m \right)^2 - 2c_{2(i)} \end{aligned}$$

To satisfy the continuity of the displacements at layer interfaces and the boundary conditions of the stress σ_z, we have

$$\begin{aligned} c_{4(i)} &= -c_{4(-i)} \\ &= \sum_{k=0}^{i-1} \left(\frac{16}{3} c_0 \left(-h_0 + \sum_{m=0}^{k} 2h_m \right)^2 \left(\frac{2 - v_k}{1 - v_k} - \frac{2 - v_{k+1}}{1 - v_{k+1}} \right) + \left(c_{3(k)} - c_{3(k+1)} \right) \left(-h_0 + \sum_{m=0}^{k} 2h_m \right) \right) \end{aligned}$$

$$c_{5(N)} = \frac{1 - 2v_N}{1 - v_N} \left(\frac{32h^3c_0}{3(1 - v_N)} + c_{3(N)}h + 2c_{2(N)}h - \frac{2v_N}{1 - 2v_N} c_{4(N)} \right)$$

$$\begin{aligned} c_{5(i)} = \frac{1 - 2v_i}{1 - v_i} \Bigg(& \frac{E_{i+1}}{E_i} \Bigg(-\frac{32}{3(1 - v_{i+1})} c_0 \left(h - \sum_{m=N}^{i+1} 2h_m \right)^3 - (c_{3(i+1)} + 2c_{2(i+1)}) \left(h - \sum_{m=N}^{i+1} 2h_m \right) \\ & + \frac{2v_{i+1}}{1 - 2v_{i+1}} c_{4(i+1)} + \frac{1 - v_{i+1}}{1 - 2v_{i+1}} c_{5(i+1)} \Bigg) + \frac{32}{3(1 - v_i)} c_0 \left(h - \sum_{m=N}^{i+1} 2h_m \right)^3 \\ & + (c_{3(i)} + 2c_{2(i)}) \left(h - \sum_{m=N}^{i+1} 2h_m \right) - \frac{2v_i}{1 - 2v_i} c_{4(i)} \Bigg) \end{aligned}$$

$$c_{5(-N)} = \frac{1 - 2v_N}{1 - v_N} \left(\frac{32h^3c_0}{3(1 - v_N)} + c_{3(-N)}h + 2c_{2(-N)}h - \frac{2v_N}{1 - 2v_N} c_{4(-N)} - \frac{q_0}{E_N} \right)$$

As a result of simplicity, $c_{5(i)}$ and $c_{5(-N)}$ can be expressed in a compact form as

$$c_{5(i)} = C_{(i)}c_0$$

$$c_{5(-N)} = Dc_0 - \frac{1-2v_N}{1-v_N}\frac{q_0}{E_N}$$

then

$$c_0 = \frac{1-2v_N}{1-v_N}\frac{q_0}{E_N}\frac{1}{D-C_{(-N)}}$$

To satisfy the continuity of the displacement w at layer interfaces, we have also

$$c_{6(i)} = \frac{8}{3}c_0\left(\sum_{m=0}^{i-1}2h_m - h_0\right)^4\left(\frac{1+v_i}{1-v_i} - \frac{1+v_{i-1}}{1-v_{i-1}}\right) + \left(\sum_{m=0}^{i-1}2h_m - h_0\right)^2\left(\frac{1}{2(1-v_i)}c_{3(i)} - \frac{1}{2(1-v_{i-1})}c_{3(i-1)} + \frac{1-2v_i}{1-v_i}c_{2(i)} - \frac{1-2v_{i-1}}{1-v_{i-1}}c_{2(i-1)}\right) + \left(\sum_{m=0}^{i-1}2h_m - h_0\right)(c_{5(i-1)} - c_{5(i)}) + c_{6(i-1)}$$

Now all of the boundary and continuity conditions are satisfied. For laminate with layers made of same material, the displacement solutions (5) are of identical formula with Eq. (3) for the axially symmetrical homogeneous isotropic circular plate.

3. "pb-2 Ritz" theory

The models of "pb-2 Ritz" theory found in publications are unfit for three-dimensional static analysis for thick circular plates, a new "pb-2 Ritz" model for thick circular plates is set up.

3.1. Single layer plates

Based on the "pb-2 Ritz" theory by Liew et al. (1998), the field of displacement is expressed as

$$u = u_0 + z\vartheta - \frac{4z^3}{3h^2}\left(\frac{\partial w_0}{\partial r} + \vartheta\right)$$

$$w = w_0$$

where

$$u_0 = \sum_{i=0}^{m} c_u^i(1-r^2)r^{2i+1}$$

$$\vartheta = \sum_{i=0}^{m} c_\vartheta^i(1-r^2)r^{2i+1}$$

$$w_0 = \sum_{i=0}^{m} c_w^i(1-r^2)^2 r^{2i}$$

It is easy to see that the clamped boundary conditions $r = 1$, $u = w = 0$ have been satisfied.

The strain energy due to bending can be expressed as

$$U = \frac{1}{2}\int_V \{\varepsilon\}^{\mathrm{T}} D\{\varepsilon\}\,\mathrm{d}V$$

The potential energy of the load distributed over the plate surface is

$$W = \int_S wq_0\,\mathrm{d}S$$

In accordance with Ritz method the total energy function can be obtained

$$\Pi = U - W$$

and the coefficients c_α^i are determined by minimizing the total energy function, i.e.

$$\frac{\partial \Pi}{\partial c_\alpha^i} = 0, \quad \alpha = u, \vartheta, w$$

3.2. Laminates

Based on the "pb-2 Ritz" theory by Liew (1994), the transverse deflection is expressed as

$$w = \sum_{i=0}^{m} c_w^i (1 - r^2)^2 r^{2i}$$

It is obvious that the clamped boundary conditions $r = 1, u = w = 0$ have been satisfied.

Following the step in Section 3.1, the coefficients c_w^i can be obtained. Wherein the stiffness coefficients D is determined as

$$D = \frac{1}{3}\sum_{k}^{2N+1} D_k\left(\left(-h_0 + \sum_{m=0}^{k} 2h_m\right)^3 - \left(-h_0 + \sum_{m=0}^{k-1} 2h_m\right)^3\right)$$

where D_k is the matrix of bending stiffness coefficients of the kth ply.

4. Experimental models and device

To verify the analysis, experiment is conducted for sandwich plates under uniform transverse load. The strains on both the surfaces and the displacement w at the top surface are tested. The models with the main dimensions presented in Table 1 are shown in Fig. 4. The faceplates are made of the same composite material with $E = 5.6$ GPa and $v = 0.15$. The core is made of rubber with $v = 0.47$ and Young's modulus E expressed as

$$\text{sample I}: \; E = (2.90 + 25.00q_0)\ \text{MPa}, \quad \text{sample II}: \; E = (6.40 + 30.00q_0)\ \text{MPa}$$

Twelve bidirectional strain gauges are glued to the top and bottom surfaces, as indicated in Fig. 5.

Table 1
The main dimensions of samples

Sample no.	Structure	R (mm)	$2H$ (mm)	$2H_0$ (mm)
I	Sandwich	100	45	41
II		100	55	41

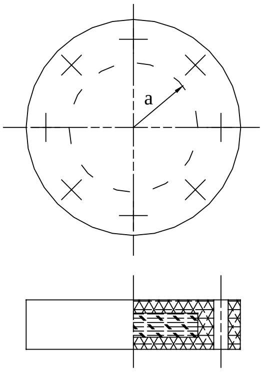

Fig. 4. Geometry of a sandwich plate.

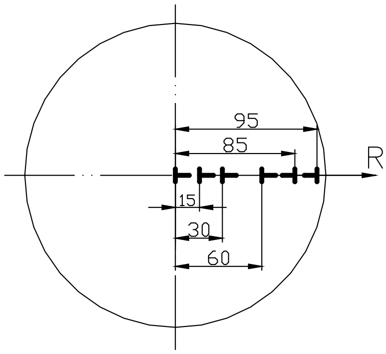

Fig. 5. The location of strain gauges (mm).

The experimental device comprises two parts: supporting structure and pressurizer, as shown in Fig. 6. The supporting structure consists of platen, sealer and pedestal. When one sample is placed between the annular packing and the annular platen, the annular platen, sample and annular packing are pressed against the pedestal by the bolts. Out-of-plane uniform pressure is supplied by the pressurizer.

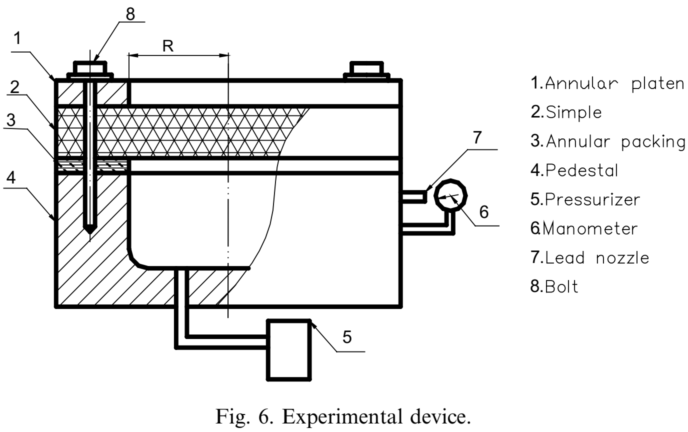

Fig. 6. Experimental device.

5. Calculation and analysis

5.1. Homogeneous plates

Tables 2 and 3 present the calculating results of the present theory (TH), "pb-2 Ritz", Ansys (ANS) and Timoshenko and Goodier (1970) three-dimensional theory (TG) for a homogeneous plate with radial $a = 100$ mm, thickness $2H = 20$ mm, and the same material as the faceplates of the samples, under uniform transverse load $q_0 = 0.12$ MPa. The element Solid45 is employed in the Ansys model.

From Table 2, it can be observed that the results of the present theory, "pb-2 Ritz" and Ansys are very close to each other except near the boundary, however, "pb-2 Ritz" theory cannot provide values of the transverse normal strain. From Table 3, it can be shown that the results of the in-plane stress σ_r of the present theory are closer to Ansys than "pb-2 Ritz". It is further shown that there exists obvious difference between the results of Timonshenko-Goodier theory (1970) and of the present theory, "pb-2 Ritz" and Ansys.

Numerical data given in Table 4 are the maximum deflections w of three homogeneous plates with the same material as mentioned above and different ratios of radius to thickness by the present theory, Ansys and "pb-2 Ritz" theory. It shows that the results of the present theory, Ansys and "pb-2 Ritz" theory are

Table 2
Single layer, $z = h$, distributions of the strains and deflection w

R (mm)	0	25	50	75	100
ε_r					
TH	183	145	36	−149	−405
ANS	170	131	34	−104	−252
Pb-2 Ritz	203	166	50	−137	−548
ε_θ					
TH	183	170	134	72	12
ANS	170	152	120	58	8
Pb-2 Ritz	203	190	153	90	0
ε_z					
TH	−44	−35	−9	33	94
ANS	−43	−31	−8	30	78
Pb-2 Ritz	–	–	–	–	–
$w \times 10^2$ (mm)					
TH	5.18	4.28	2.76	0.99	0.00
ANS	5.02	4.43	2.91	1.11	0
Pb-2-Ritz	5.81	5.14	3.40	1.28	0

Table 3
Single layer, $z = h$, distribution of the stress

R (mm)	0	20	40	60	80	100
σ_r (MPa)						
TH	1.21	1.07	0.64	−0.07	−1.06	−2.33
ANS	1.23	0.99	0.65	0.03	−0.83	−2.35
Pb-2 Ritz	1.41	1.26	0.82	0.03	−1.02	−3.24
TG	−0.01	−0.15	−0.58	−1.29	−2.28	−3.56

Table 4
Single layer, $z = h$, $R = 0$, the deflection w

a/H	40	20	10	7
$w \times 10^2$ (mm)				
TH	331.8	41.47	5.18	1.78
ANS	310.8	37.16	5.02	1.75
Pb-2 Ritz	328.2	42.12	5.81	2.24

Table 5
Sandwich plate the experimental, present theoretical and Ansys's results of strains

q_0 (MPa)	0.02						0.12					
R (mm)	0	15	30	60	85	95	0	15	30	60	85	95
Sample I												
ε_r												
$z = h$												
EX	335	351	289	20	−488	−617	1431	1633	1609	728	−1920	−2796
TH	359	327	251	−29	−429	−623	1375	1250	962	−110	−1637	−2381
$Z = -h$												
EX	−299	−300	−246	−67	200	693	−816	−824	−744	−100	1181	3305
TH	−367	−334	−257	29	437	637	−1402	−1275	−981	113	1671	2428
ε_θ												
$z = h$												
EX	319	299	279	209	75	−294	1412	1394	1283	926	278	−1376
TH	359	346	323	230	97	32	1375	1333	1287	879	371	123
$Z = -h$												
EX	−338	−320	−280	−214	−49	−24	−925	−922	−927	−950	−394	−224
TH	−367	−356	−330	−235	−99	−33	−1402	−1360	−1261	−897	−388	−124
Sample II												
ε_r												
$z = h$												
EX	108	98	68	7	−75	−105	585	534	374	38	−395	−571
TH	107	97	74	−9	−129	−187	614	557	428	−54	−721	−1095
ANS	76	64	45	−10	−75	−106	438	376	182	−54	−452	−642
$Z = -h$												
EX	−119	−120	−74	−9	106	164	−682	−672	−407	−21	637	972
TH	−120	−102	−83	10	144	210	−669	−608	−466	59	807	1172
ANS	−70	−82	−75	−3	118	182	−346	−436	−412	−26	642	998
ε_θ												
$z = h$												
EX	102	98	88	64	33	−30	553	335	479	350	172	−166
TH	107	103	96	68	27	10	614	594	552	391	164	48
ANS	70	68	63	44	18	6	408	399	371	260	110	36
$Z = -h$												
EX	−113	−115	−107	−80	−41	−13	−627	−651	−620	−476	−242	−65
TH	−119	−116	−107	−76	−32	−10	−669	−648	−601	−426	−177	−55
ANS	−77	−80	−79	−61	−30	−13	−398	−422	−418	−4326	−163	−64

very close and when the ratio of radius to thickness decreases to 10, the results of the present theory is closer to Ansys.

Hence, it can be concluded that the theory of homogeneous plates is reliable and new.

5.2. Sandwich plates

Numerical data given in Tables 5 and 6 are the results of the experiment (EX), the present sandwich theory and Ansys for the sandwich plates shown in Table 1.Taking into account the ratio of the core thickness to the total thickness, Shell91 and Solid45 are employed for sample I and sample II respectively.

Table 6
Sandwich plate $z = h$ the experimental, present theoretical and Ansys's results of displacement w

q_0 (MPa)	0.02								0.12							
R (mm)	16	25	40	55	60	75	80	95	16	25	40	55	60	75	80	95
$w \times 10^2$ *(mm)*																
Sample I																
EX	68	–	56	–	34	–	9	2	266	–	217	–	141	–	40	2
TH	84		73		36		9	2	322		242		141		44	11
ANS	103		89		68		38	10	358		308		235		132	35
Sample II																
EX	–	6.8	5.8	4.3	–	1.8	–	0.3	–	38.1	32.0	23.2	–	12.0	–	4.0
TH		6.6	5.4	3.5		1.5		0.3		38.1	31.0	21.1		8.7		1.3
NS		4.8	3.8	2.6		1.0		0.1		28.2	22.1	15.3		5.0		0.3

Table 7
Sample II $R = 0$ $q_0 = 0.02$ MPa the present theoretical and Ansys's results of stresses

Z (mm)	−27.5	−22	−20.5	−11	−5.5	0	5.5	11	20.5	22	27.5
σ_r (MPa)											
TH	−0.794	0.448	0.790	−0.011	−0.010	−0.008	−0.007	−0.005	−0.711	−0.403	0.705
ANS	−0.507	0.332	0.314	−0.010	−0.009	−0.008	−0.008	−0.008	−0.233	−0.264	0.440
σ_θ (MPa)											
TH	−0.794	0.448	0.790	−0.011	−0.010	−0.008	−0.007	−0.005	−0.711	−0.403	0.705
ANS	−0.507	0.332	0.314	−0.010	−0.009	−0.008	−0.008	−0.008	−0.233	−0.264	0.440
σ_z (MPa)											
TH	−0.020	−0.012	−0.011	−0.011	−0.010	−0.010	−0.009	−0.008	−0.008	−0.007	0.000
ANS	−0.017	−0.020	−0.016	−0.014	−0.014	−0.013	−0.012	−0.011	−0.009	−0.003	−0.003

Table 8
Sandwich plate $z = h$ the present theoretical and Ansys's results of displacement w

R (mm)	0	20	40	60	80	100
$w \times 10^2$ *(mm)*						
$\alpha = 100$						
TH	20.2	18.7	14.3	8.30	2.60	0.00
ANS	18.0	17.3	15.0	11.4	6.32	0.00
$\alpha = 216$						
TH	43.0	39.6	30.4	17.6	5.58	0.00
ANS	38.1	36.5	31.9	24.2	13.5	0.00
$\alpha = 1080$						
TH	189	175	134	78	25	0
ANS	187	180	157	119	67.3	0

From the data in Tables 5 and 6, it can be seen that the results of the present theory are closer to the experiment than Ansys. The displacement values of the present theory are between Ansys and experiment. For the distribution of the displacement and the strains, the present theory are in agreement with the experiment: the maximum of normal strain ε_x exists on the boundary, of normal strain ε_y exists at the center of the circular plate; the displacement distributions in the r-direction are all in cosine type but Shell91, their maximums exist at the center.

The data given in Table 7 are the present theoretical, Ansys's reaults of stresses. It is clear that the stresses in the core are almost zero due to its moduli very small in comparison with the faceplates. Similar to the difference between exact solutions and a $\{3,2\}$-order theory by Barut et al. (2001), difference between the results of the present theory and Ansys exists.

The deflection w distributions of the theory are compared with Ansys for sandwich plates with the same structure as sample I and different α, the ratio of Young's modulus of the faceplates and the core, as shown

Table 9
Laminate $z = -h$ the present theoretical and Ansys's results of the strains and deflection w "pb−2 Ritz" theoretical results of the deflection w

R (mm)	0	20	40	60	80	100
ε_r						
TH	−174	−153	−87.4	−21.5	174	370
ANS	−168	−140	−84.4	12.6	146	330
ε_θ						
TH	−174	−167	−146	−109	−58.3	6.94
ANS	−154	−148	−143	−98.7	−63.1	−3.76
ε_z						
TH	51.1	46.0	30.6	4.96	−30.9	−77.0
ANS	47.3	41.4	33.9	3.00	−22.6	−22.5
$w \times 10^2$ (mm)						
TH	6.03	5.56	4.26	2.48	0.80	0.03
ANS	6.29	5.83	4.52	2.71	0.94	0.00
Pb-2 Ritz	6.04	5.57	4.27	2.48	0.78	0.00

Table 10
Laminate $R = 50$ mm the present theoretical and Ansys's results of the strains

z (mm)	−7.5	−4.5	−1.5	1.5	4.5	7.5
ε_r						
TH	−47.5	−27.3	−8.83	8.82	27.3	47.5
ANS	−48.3	−28.7	−9.06	8.93	29.2	45.0
ε_θ						
TH	−132	−78.1	−25.8	25.8	78.1	132
ANS	−118	−73.0	−23.3	27.5	75.5	116
ε_z						
TH	21.2	9.16	−0.1	−11.0	−19.7	−31.7
ANS	21.5	9.68	−1.35	−11.0	−20.1	−29.9
γ_{rz}						
TH	0.00	−39.6	−64.3	−64.9	−39.6	0.00
ANS	−17.6	−37.5	−59.6	−58.8	−38.4	−19.1

Table 11
Laminate $z = h$, $q_0 = 0.12$ MPa the deflections w

R (mm)	0	20	40	60	80	100
$w \times 10^2$ *(mm)*						
$\alpha = 1.3$						
TH	6.0	5.5	4.2	2.5	0.8	0.0
ANS	6.2	5.8	4.5	2.7	0.9	0.0
"pb−2 Ritz"	6.0	5.6	4.3	2.5	0.8	0
$\alpha = 2.0$						
TH	6.1	5.6	4.3	2.5	0.8	0.0
ANS	6.5	6.0	4.7	2.8	1.0	0.0
"pb-2 Ritz"	6.1	5.6	4.2	2.5	0.8	0.0
$\alpha = 8.0$						
TH	6.2	5.7	4.4	2.5	0.8	0.0
ANS	8.1	7.6	6.0	3.8	1.4	0.0
"pb-2 Ritz"	6.2	5.7	4.4	2.5	0.8	0.0

in Table 8. It is clear that the maximums of deflection w of the theory coincides with Ansys nicely, but with the increasing of α, the difference near the boundary increases. The w of Ansys are in parabola type, but of the present theory in cosine type. It is evident that this reflects a discrepancy between three-dimensional analysis and two-dimensional analysis. Hence, we can arrive at a conclusion that the trend is reasonable.

5.3. Laminates

The laminate under uniform transverse load $q_0 = 0.12$ MPa consists of three layers of the same thickness $2H_i = 5$ mm and radius $a = 100$ mm including two faceplates with $E = 1.08 \times 10^4$ MPa, $v = 0.15$ and a middle layer with $E = 1.00 \times 10^4$ MPa, $v = 0.17$. The results of the present theory, "pb-2 Ritz" and Ansys are shown in Tables 9 and 10. Solid45 element of Ansys is used for calculation. From these tables, it is clear that the present theory, "pb-2 Ritz" and Ansys are close to each other.

For different ratio of modulus of the faceplates and the middle, The deflection w of the present theory, Ansys and "pb-2 Ritz" theory for laminates with the total thickness $2H = 20$ mm, radius $R = 100$ mm and three plies of same thickness are shown in Table 11. Solid45 element is used for Ansys calculation.

From Table 11, it can be seen that with the ratio increasing the discrepancy between the theories and Ansys increases, but the present theory and "pb-2 Ritz" theory are in agreement with each other.

6. Conclusions

The paper presents specifically the solutions for axially symmetrical homogeneous isotropic circular plates, laminate plates and sandwich plates under transverse load. The solutions including all of normal stresses and shear stresses satisfy all the equations of equilibrium, the mid-plane clamped boundary conditions and the interfacial continuity conditions through-thickness. Comparing with experimental, Ansys's and "pb-2 Ritz" theoretical results, it can be concluded as follows:

1. The solution for homogeneous plates are different from the solution by Timoshenko and Goodier (1970), it is a new three-dimensional theoretical solution.

2. Within a scope of ratio of radius to thickness, the clamped boundary conditions can be accurately simplified as the mid-plane clamped boundary conditions, the present solution for single layer plate is precise solution for axially symmetrical homogeneous isotropic circular plates under uniform transverse load with clamped boundary.
3. The present sandwich theory is in agreement with the experiment, so it can be used for estimating the distribution of strains and stresses of the faceplate.
4. The present laminate theory agrees with "pb-2 Ritz" theory, and within a scope of ratio of radius to thickness, is in agreement with Ansys, thus the clamped boundary conditions can be accurately simplified as the mid-plane clamped boundary conditions.

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